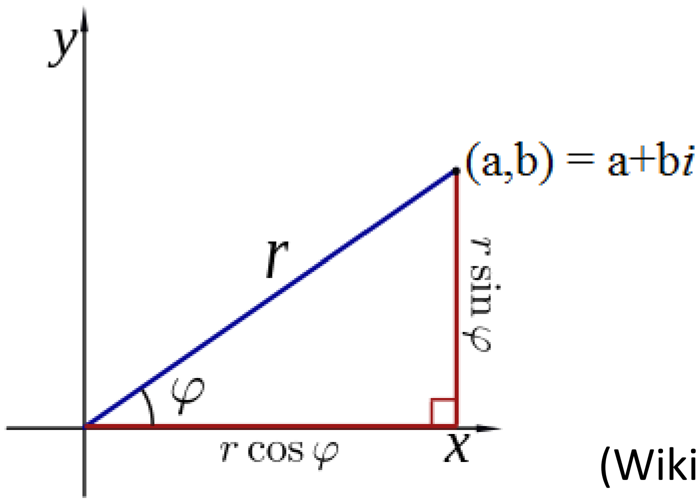
Linear Algebra & Geometry

LECTURE 2

Complex numbers cont.

A point *z* of the plane can be identified by its Cartesian coordinates, say (a,b), but also by its *polar coordinates*, i.e. the distance *r* from the origin and the angle φ between positive half-axis OX and the segment (0,0)(a,b). Hence, (a,b) = (r*cos* φ,r*sin* φ) or, equivalently, z = a+b*i* = r(*cos* φ+ i*sin* φ). Clearly, r =$\sqrt{a^2+b^2}$, i.e. r = |z|



(Wikipedia & TT)

Definition.

The formula r(*cos* φ+ i*sin* φ) is known as the *polar form* (sometimes *trigonometric form*) of the complex number z.

Remarks.

- $(-1)(\cos\varphi + i\sin\varphi)$ is NOT a polar form of a complex number
- $7(\cos\alpha + i\sin\varphi)$ is NOT a polar form (unless $\alpha = \varphi$)
- $666(\cos\varphi - i\sin\varphi)$ is NOT a polar form (unless $\sin\varphi = 0$)
- The question "where the hell is this imaginary unit *i*" suddenly becomes meaningful. The answer is "at (0,1)".

Comprehension

1. Find the polar form of 1, -1, i and $-i$.
2. Find the polar form of $666(\cos\alpha - i\sin\alpha)$
3. Knowing that the polar form of z is $r(\cos\alpha + i\sin\alpha)$ find the polar form of $\bar{z}$.

The angle φ is called an argument of z. Since both sine and cosine are periodic function with the period of 2π, a complex number has infinitely many arguments, each of the form $\varphi+2k\pi$ for some integer k. Hence the term "THE polar form of z" is a slight abuse of language.

Definition.
The argument of z belonging to the interval <0;2 π) is called the *principal argument* of z.

Example.
The polar form of z=1+i is $\sqrt{2}(cos\frac{\pi}{4}+i\ sin\frac{\pi}{4})$, the polar form of z=1 is $cos0+isin0$, for z=-1 is $\cos\pi+i\sin\pi$

It helps if you memorize values of sine and cosine for those basic angles 0, $\frac{\pi}{4},\frac{\pi}{3},\frac{\pi}{6},\frac{\pi}{2}$ and the like.

Theorem (Multiplication Lemma)

Let z = $r(\cos \alpha + i \sin \alpha)$ and $w = t(\cos \varphi + i \sin \varphi)$ be two complex numbers. Then

$$zw = rt(\cos(\alpha+\varphi) + i \sin(\alpha+\varphi)).$$

Proof.

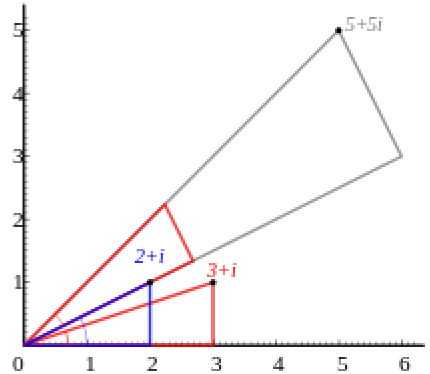
$zw = r(\cos \alpha + i \sin \alpha)t(\cos \varphi + i \sin \varphi) = rt((\cos \alpha \cos \varphi - \sin \alpha \sin \varphi) + i(\cos \alpha \sin \varphi + \sin \alpha \cos \varphi)) = rt(\cos(\alpha+\varphi) + i \sin(\alpha+\varphi))$.

The last transformation follows from well-known trigonometric identities. QED

Remark.

Another way of representing a complex number $z = r(\cos \alpha + i \sin \alpha)$ is the *exponential form* $z = re^{i\alpha}$. By laws of exponentiation we obtain a similar law: $zw = rt\, e^{i(\alpha+\varphi)}$.

Remark. This is as close as we can get to a geometrical interpretation of complex multiplication: when you multiply *z* by *w* you rotate the vector representing *z* counterclockwise by the argument of *w* and you adjust the length so that it becomes the product of lengths of *z* and *w.*



Picture from Wikipedia

Corollary (of Multiplication Lemma)

Let z = $r(\cos\alpha + i\sin\alpha)$ and $w = t(\cos\varphi + i\sin\varphi)$ be two complex numbers. Then

$$\frac{z}{w} = \frac{r}{t}(\cos(\alpha - \varphi) + i\sin(\alpha - \varphi)).$$

Proof.

$\frac{z}{w}$ is the only number x satisfying $xw = z$.

Try $x = \frac{r}{t}(\cos(\alpha - \varphi) + i\sin(\alpha - \varphi))$.

Using the Multiplication Lemma we obtain

$xw = \frac{r}{t}(\cos(\alpha - \varphi) + i\sin(\alpha - \varphi)) \cdot t(\cos\varphi + i\sin\varphi) =$

$r(\cos((\alpha - \varphi)+\varphi) + i\sin((\alpha - \varphi)+\varphi)) = r(\cos\alpha + i\sin\alpha) =$

z hence, $x = \frac{r}{t}(\cos(\alpha - \varphi) + i\sin(\alpha - \varphi)) = \frac{z}{w}$. QED

Corollary (de Moivre Law)

Let z = $r(\cos \alpha + i \sin \alpha)$. Then for every positive integer n

$$z^n = r^n(\cos n\alpha + i \sin n\alpha).$$

Proof.
The formula follows from a repeated application of the Multiplication Lemma. (Use induction if you want to be VERY rigorous). QED

Remark. This means that when you raise z of modulus 1 to n-th power, geometrically you rotate z counterclockwise n-1 times by α (the modulus stays 1).

Example.

Calculate z^{10} where $z = 1 + i\sqrt{3}$. We will use de Moivre Law. First, we find the modulus of z and factor it out. Since $|z| = 2$ we can write $z = 2(\frac{1}{2} + i\,\frac{\sqrt{3}}{2})$. The number in parenthesis belongs to the unit circle, hence, there exists α such that $\cos\alpha = \frac{1}{2}$ and $sin\alpha = \frac{\sqrt{3}}{2}$. If you recall your high school algebra the angle is $\frac{\pi}{3}$, i.e. $z=2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ and $z^{10} = 2^{10}(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}) = 1024\,(\cos(2\pi + \frac{4\pi}{3}) + i\sin(2\pi + \frac{4\pi}{3})) = 1024\,(\cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3})) = 1024(-\frac{1}{2} + i(-\frac{\sqrt{3}}{2}))$.

This is much more fun than going $(1 + i\sqrt{3})\,(1 + i\sqrt{3})\,\ldots$ ten times.

Definition 1.2.
Every complex number w satisfying the equation w^n=z is called a *root of z of order n*.

Notice that, unlike in real numbers, both -2 and 2 are called square roots of 4. De Moivre Law can be used to calculate complex roots.

Root formula
Take z=r(cos α+$i$sin α) and suppose w=p(cosφ +$i$sin φ) is a root of z of order n. Then $w^n = p^n(cos\ n\varphi + i\ sin\ n\varphi) = r(cos\ \alpha + i sin\ \alpha)$. Hence, $p= \sqrt[n]{r}$ (in the usual, real-number sense) and $cos\ n\varphi = cos\ \alpha$ and $sin\ n\varphi = sin\ \alpha$.

Since 2π is the period of both sine and cosine, we obtain

$n\varphi_k = \alpha + k2\pi$ or, equivalently, $\varphi_k = \frac{\alpha+2k\pi}{n}$, for k=0,±1, ± 2, ...

So, $w_k = \sqrt[n]{r}(cos\frac{\alpha+2k\pi}{n} + i\sin\frac{\alpha+2k\pi}{n})$.

Consider two roots whose indices differ by n, say w_k and w_{k+n}.

$$w_{k+n} = \sqrt[n]{r}\left(cos\,\frac{\alpha+2(k+n)\pi}{n} + i\sin\frac{\alpha+2(k+n)\pi}{n}\right) =$$

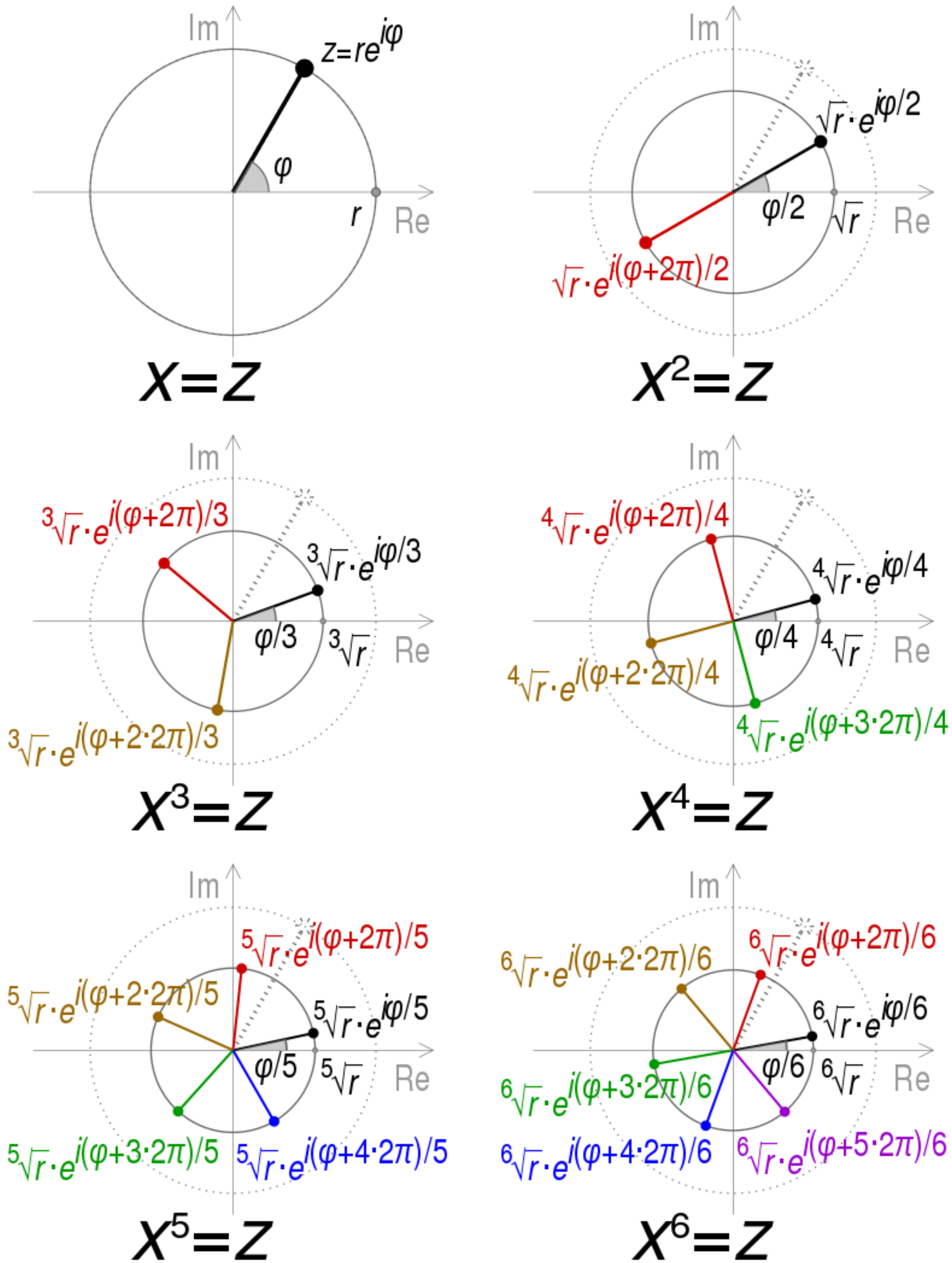
$$\sqrt[n]{r}\left(cos\,\frac{\alpha+2k\pi+2n\pi}{n} + i\sin\frac{\alpha+2k\pi+2n\pi}{n}\right) =$$

$$\sqrt[n]{r}\left(\cos\left(\frac{\alpha+2k\pi}{n} + 2\pi\right) + i\sin\left(\frac{\alpha+2k\pi}{n} + 2\pi\right)\right) =$$

$$\sqrt[n]{r}\left(\cos\left(\frac{\alpha+2k\pi}{n}\right) + i\sin\left(\frac{\alpha+2k\pi}{n}\right)\right) = w_k.$$

This indicates that we only get n different roots of z of order n, namely $w_0, w_1, \ldots, w_{n-1}$ – no more, no less.

Important. The root formula and its consequences apply only to roots of a number **not to roots of a polynomial**.



Picture from Wikipedia.

Roots of order n of a complex number z are uniformly distributed over the circle centered at 0 and with the radius of $\sqrt[n]{r}$. The angular distance between any two consecutive roots is $\frac{2\pi}{n}$.

Polynomials

Definition. A *polynomial of degree n* over a set K is any function f: K→K of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n \in K$ and $a_n \neq 0$. The set of all polynomials over K is denoted by K[x].

We adopt the convention that the degree of the zero polynomial 0 is $= -\infty$. For other constant polynomials, the degree is 0. K will usually be the set $\mathbb{R}$ or $\mathbb{C}$.

Definition.
Let f be a polynomial over K. A number a from K is called a *root* of f if and only if $f(a) = 0$.

Fact. (Remainder lemma)
For every two polynomials f, $g \in K[x]$ with $g \neq 0$ there exist polynomials q, $r \in K[x]$ such that

$$f = qg + r \text{ and } deg(r) < deg(g).$$

Corollary.

A number a is a root of a polynomial f(x) if and only if f(x) is divisible by $(x - a)$.

Theorem (Main Theorem of Algebra)

Every polynomial $f \in \mathbb{C}[x]$ of degree at least 1 has a root in $\mathbb{C}$.

Corollary.

Every polynomial $f \in \mathbb{C}[x]$ of degree n has exactly n roots (counting multiplicities).

Theorem.
If $f \in \mathbb{R}[x]$ then for every root z of f, $\overline{z}$ is also a root of f.
Proof.
$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$. Hence, $\overline{f(z)} = \overline{0} = 0$, and

$$\overline{f(z)} = \overline{a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0} =$$

$$\overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \ldots \overline{a_1 z} + \overline{a_0} =$$

$$a_n \overline{z}^n + a_{n-1} \overline{z}^{\,n-1} + \ldots + a_1 \overline{z} + a_0 = f(\overline{z}) = 0.$$

Corollary.
If $f \in \mathbb{R}[x]$ then f can be factored into a product of polynomials from $\mathbb{R}[x]$ of degree at most 2 each.
Proof.
It follows from the fact that $\big(x-(a+bi)\big)\big(x-(a-bi)\big) =$
$x^2 - x(a-bi) - x(a+bi) + (a+bi)(a-bi) =$
$x^2 - 2ax + xbi - xbi + a^2 + b^2.$

Comprehension.

Prove on your own: every polynomial from $\mathbb{R}[x]$ with on odd degree has at least one real root.